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2-Generalized burning number of square graphs

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Abstract

The burning number of a graph G, denoted by $b(G)$, is the minimum number of steps it takes to burn the graph. 2-Generalized burning number of G which is a generalization of $b(G)$, denoted by $b_2(G)$, is the minimum number of steps it takes to burn every vertex of G by burning vertices only if they are adjacent to at least 2 burned neighbors. In this paper, we give some bounds for the 2-generalized burning number of square graphs. In particular 2-generalized burning number of square graphs for some specific graphs are obtained.

Keywords: 2-generalized burning number, Square graph.

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1. Introduction

A graph G is a pair $G = (V, E)$, where V is the set of vertices of G and E is a subset of $P_2(V)$ called the edge set of G. We say two vertices u and v in G are adjacent if $(u, v) \in E$. The degree of a vertex v in G, denoted by $\deg(v)$, is the number of vertices adjacent to v. A graph G is r- regular if every vertex of G has degree r. A path is a sequence of vertices, such that there is an edge from each vertex to the next vertex of this sequence. A path graph P_n is a graph whose vertices can be listed in the order $\{v_1, v_2, \cdots, v_n\}$ such that the edges are $e_i = (v_i, v_{i+1})$, where $i = 1, 2, \cdots, n-1$. A Hamiltonian path is a path in a graph that visits each vertex exactly once. The distance between two vertices u and v of G, denoted by $d_G(u, v)$, is length of the shortest path between u and v. The eccentricity of a vertex v is defined as $max\{d_G(v, u) : u \in V(G)\}$. Radius and diameter of G is the minimum and maximum eccentricity over the set of all vertices of G, denoted by $r(G)$ and $d(G)$, respectively. A subgraph H of a graph G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A spanning subgraph H of G is a subgraph of G with $V(H) = V(G)$. The needed terminologies are from [4].

A complete graph is a graph in which each pair of vertices is connected by an edge. A complete graph with n vertices is denoted by K_n. A simple graph with one vertex of degree $n-1$ and $n-1$ vertices of degree 1 is called a star graph denoted by S_n. A strongly regular graph with parameters (n, k, λ, μ) (for short, a $srg(n, k, \lambda, \mu)$) is a k-regular graph on n vertices such that any two adjacent vertices have

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exactly λ common neighbours and any two nonadjacent vertices have exactly μ common neighbours[4]. The square graph G^2 of G is the graph with vertex set $V(G)$ such that two vertices u and v are adjacent if and only if $d_G(u,v) \leqslant 2$. A set $S \subseteq V(G)$ is called a 2-dominating set if every vertex not is S has at least two neighbors in S. Let G be a connected graph, the process of burning G begins with any vertices being unburned. In the first step, a single vertex is selected to be burned. In every subsequent time step, either the fire spreads to all neighbours of a previously burned vertex, and those vertices become burned, or another vertex is selected to be burned. This means that if a vertex is already burned at a step, then its unburned neighbors (if any) become automatically burned in next step. When all the vertices are burned, the process ends. The minimum number of time steps needed to burn all the vertices of a graph is named the burning number of G and denoted by $b(G)$ [2]. 2-generalized burning number of G, denoted by $b_2(G)$, is the minimum number of steps it takes to burn every vertex of by burning vertices only if they are adjacent to at least 2 burned neighbors. we call this discrete time process as a 2-burning process of G. If all vertices of the graph are burned after k times steps by 2-burning process, we call sequence $(x_1, x_2, \ldots, x_k)$ a 2-burning sequence of graph. The 2- generalized burning number of G is defined as the length of a minimum burning sequence among all 2-burning process of G [3].
To give our results first we remind some needed theorems:

Theorem 1.1. *[3] If G is a path graph of order n, or is a circle graph of order $n \geqslant 3$, then:*

$$b_2(G) = \lceil \tfrac{n}{2} \rceil + 1.$$

Theorem 1.2. *[3] Suppose G is a complete graph of order $n \geqslant 2$, then:*

$$b_2(G) = 3.$$

2. Main Results

In what follows all graphs are simple and connected. We give some bounds for the 2-generalized burning number in square graphs, and obtain the 2-generalized burning number in the square graph of some specific graphs.

Lemma 2.1. *Let G be a connected graph and S be a 2-dominating set for G, then $b_2(G) \leqslant \mid S \mid +1$.*

Proof. Since S is a 2- dominating set for G, first we burn all vertices of S then burn a unburned vertex which implies to burning all vertices of G. □

Theorem 2.2. *Let G be a graph of order n with a hamiltonian path, then $b_2(G^2) \leqslant [\frac{n}{3}] + 2$.*

Proof. Let the ordered n-tuple $(v_1, v_2, \ldots, v_n)$ denotes the vertex order of a hamiltonian path in G. If $n = 3k$, then set $S = \{v_1, v_4, \ldots, v_{3k-2}, v_{3k}\}$ and If $n = 3k+1$, the set $S = \{v_1, v_4, \ldots, v_{3k-2}, v_{3k+1}\}$ is a 2-dominating set for G^2. Also if $n = 3k+2$, the set $S = \{v_1, v_4, \ldots, v_{3k-2}, v_{3k+1},\}$ is a 2-dominating set for $G^2 - (v_{3k+2})$. Therefore, according to lemma 2.1 we have $b_2(G^2) \leqslant \mid S \mid +1 = [\frac{n}{3}] + 2$.

□

Lemma 2.3. *Let G be a connected graph, then $b_2(G^2) \leqslant b_2(G)$.*

Proof. Clearly G is an spanning subgraph of G^2 and $\mid E(G) \mid \leqslant \mid E(G^2) \mid$, hence $b_2(G^2) \leqslant b_2(G)$.

□

Theorem 2.4. *Let $n = 3k + i$ $(0 \leqslant i \leqslant 2)$, then $b_2({P_n}^2) = k + 2$.*

Proof. Since ${P_n}^2$ is a hamiltonian graph by theorem 2.2, $b_2({P_n}^2) \leqslant [\frac{n}{3}] + 2 = k + 2$. For the other side let $\{x_1, x_2, \ldots, x_b\}$ be a 2- burning sequence for ${P_n}^2$ with $b \leqslant k + 1$. Suppose the vertex x_i $(1 \leqslant i \leqslant b - 1)$ burn in step i then in next step at most 2 vertices are added to previously burned vertices, so the maximum number of burned vertices in an optimal 2- burning sequence is $5 + 3(b - 3)$. Since $b \leqslant k + 1$, at most $5 + 3(b - 3) \leqslant 5 + 3(k + 1 - 3) = 3k - 1$ vertices are burned, this contradicts $n = 3k + i$. Thus $b_2({P_n}^2) = k + 2$.

□

Similar to the above theorem we have:

Theorem 2.5. *If* $n = 3k + i$ *and* $0 \leqslant i \leqslant 2$, *then* $b_2({C_n}^2) = k + 2$.

Let W_n be the wheel graph with n vertices. The Helm graph denoted by H_n is a graph defined by adjoining a vertex of degree one to any vertex of the cycle of a wheel [1].

Theorem 2.6.
- *i. If* G= *srg*(n, k, λ, μ), *then* $b_2(G^2) = 3$.
- *ii. If* $G = H_n$ *and* $n \geqslant 3$, *then* $b_2(G^2) = 4$.

Proof. *i*. Since G^2 is a complete graph by theorem 1.2 we have $b_2(G^2) = 3$.
ii. The graph G^2 contains a subgraph ${W_n}^2$ whose vertices are a 2- dominating set for G^2. By lemma 2.1 and the previous theorem we have

$$b_2(G^2) \leqslant b_2({W_n}^2) + 1 = 3 + 1 = 4.$$

It is easy to see that no 2- burning sequence of length 3 does burn G^2. So $b_2(G^2) = 4$. □

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